Supplementary Material of "An Efficient Evolutionary Algorithm for Few-for-Many Optimization"

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S-I. Existing F4M Test Problems

A. DC-MaTS Test Problems [1]

The general formulation for all DC-MaTS test problems is given by:

$$f_i(\boldsymbol{x}) = -g(x_i) + \frac{1}{d-1}\left(-g(x_i) + \sum_{a=1}^{d} g(x_a)\right), \quad i = 1, 2, \ldots, m,$$

where $\boldsymbol{x} = [x_1, x_2, \ldots, x_d]^\top \in [0,1]^d$ is the decision vector, $d \geq m$ is the problem dimension, and $g(x)$ is a monotonically increasing function satisfying $g(0) = 0$ and $g(1) = 1$. Under this formulation, the optimal solution to each objective f_i is achieved when $x_i = 1$ and all other variables are set to 0.

Four specific instances are defined in the DC-MaTS suite by choosing different forms for the function $g(x)$:

- **DC-MaTS1**: $g(x) = x$ (linear function).
- **DC-MaTS2**: $g(x) = x^2$ (quadratic function).
- **DC-MaTS3**: $g(x) = \sqrt{x}$ (square root function).
- **DC-MaTS4**: $g(x) = \sin(0.5\pi x)$ (sine function).

In the experiments, we set $d = m$ to maintain consistency between the number of decision variables and objectives.

B. Noisy Mixed Linear Regression [2]

The **Noisy Mixed Linear Regression (NMLR)** test problem is designed to evaluate the performance of optimization algorithms under the setting of mixed data sources with additive Gaussian noise. This problem originates from a multi-objective learning scenario where each objective corresponds to fitting a distinct noisy linear model.

The i-th objective function is defined as:

$$f_i(\boldsymbol{x}) = \frac{1}{2}\left(\langle \boldsymbol{a}^{(i)}, \boldsymbol{x}\rangle - b^{(i)}\right)^2 + \frac{\lambda}{2}\|\boldsymbol{x}\|^2,$$

where $(\boldsymbol{a}^{(i)}, b^{(i)})$ represents the i-th data point and $\lambda = 0.01$ is a fixed ℓ_2 regularization parameter. The data are generated as follows:

- Sample k ground-truth model parameters $\{\hat{\boldsymbol{x}}^{(i)}\}_{i=1}^{k}$ from a uniform distribution: $\hat{\boldsymbol{x}}^{(i)} \sim \mathrm{Uniform}(-\mathbf{1}, \mathbf{1})$.
- Sample m data points $\boldsymbol{a}^{(i)} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{I}_d)$ independently.
- Assign each $\boldsymbol{a}^{(i)}$ a random class index $c_i \sim \mathrm{Uniform}([k])$.
- Generate additive noise $\epsilon_i \sim \mathcal{N}(0, \sigma^2)$ where σ is set to 0.1.
- Construct $b^{(i)} = \langle \boldsymbol{a}^{(i)}, \hat{\boldsymbol{x}}^{(c_i)}\rangle + \epsilon_i$.

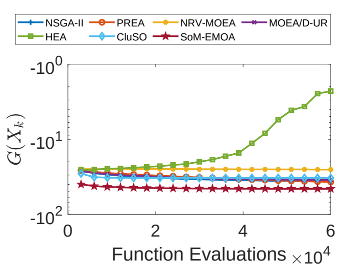
The goal is to find k parameter vectors $\{\boldsymbol{x}^{(i)}\}_{i=1}^{k}$ that together minimize the set of m objective functions. The dimension d of the decision variable $\boldsymbol{x}$ is set to 10 in our experiments.

This setup simulates a realistic scenario where each regression task originates from a different ground truth model with stochastic perturbations, thereby providing a challenging testbed for optimization under heterogeneous and noisy conditions.

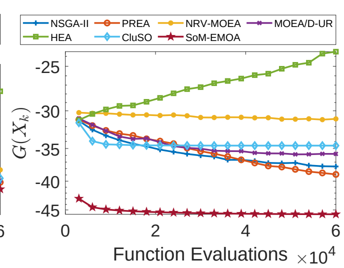
S-II. Experimental Results for $k = 10$

TABLE S-I
THE MEAN VALUES OF $G_{\text{WS}}(X_k)$ OBTAINED BY SOM-EMOA AND ALL BASELINES ON EXISTING BENCHMARK TEST PROBLEMS. THE SIZE OF THE FINAL SOLUTION SET k IS 10. THE '+', '−' AND '≈' INDICATE THAT THE COMPARED ALGORITHM IS 'SIGNIFICANTLY BETTER THAN', 'SIGNIFICANTLY WORSE THAN' AND 'STATISTICALLY SIMILAR TO' SOM-EMOA, RESPECTIVELY.

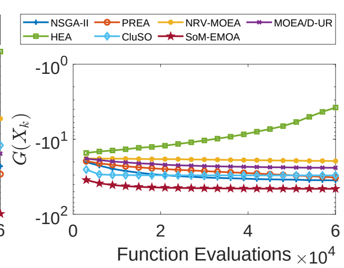
Problem	m	d	NSGA-II	PREA	NRV-MOEA	MOEA/D-UR	HEA	CluSO	SoM-EMOA
DC-MaTS1	25	25	-2.0218e+1 −	-1.8493e+1 −	-4.4577e+0 −	-1.8710e+1 −	-1.2155e+1 −	-1.7541e+1 −	-2.3328e+1
	50	50	-3.6432e+1 −	-3.2829e+1 −	-2.2846e+0 −	-3.7129e+1 −	-2.5355e+1 −	-3.5016e+1 −	-4.5790e+1
	75	75	-5.1064e+1 −	-4.9973e+1 −	-3.1314e+1 −	-5.5111e+1 −	-3.6421e+1 −	-4.9826e+1 −	-6.5956e+1
	100	100	-6.4968e+1 −	-6.8615e+1 −	-4.4582e+1 −	-6.7612e+1 −	-4.7121e+1 −	-6.3497e+1 −	-8.3230e+1
DC-MaTS2	25	25	-2.0446e+1 −	-1.9473e+1 −	-8.2020e+0 −	-1.8901e+1 −	-1.4094e+1 −	-1.9663e+1 −	-2.3328e+1
	50	50	-3.7669e+1 −	-3.4589e+1 −	-2.3552e+1 −	-3.8916e+1 −	-3.1006e+1 −	-3.5782e+1 −	-4.5800e+1
	75	75	-5.3825e+1 −	-5.1869e+1 −	-4.1244e+1 −	-5.7718e+1 −	-4.4339e+1 −	-5.0118e+1 −	-6.7279e+1
	100	100	-6.8756e+1 −	-6.8913e+1 −	-5.5129e+1 −	-7.1717e+1 −	-5.7599e+1 −	-6.4518e+1 −	-8.7770e+1
DC-MaTS3	25	25	-1.9900e+1 −	-1.6934e+1 −	-3.7221e+0 −	-1.7564e+1 −	-1.1900e+1 −	-1.5845e+1 −	-2.3331e+1
	50	50	-3.5345e+1 −	-3.0333e+1 −	-3.7780e+0 −	-3.2307e+1 −	-1.9524e+1 −	-2.4006e+1 −	-4.5754e+1
	75	75	-4.8884e+1 −	-4.2882e+1 −	-2.0377e+1 −	-4.9403e+1 −	-2.6604e+1 −	-3.7024e+1 −	-6.4797e+1
	100	100	-6.0687e+1 −	-5.6890e+1 −	-3.1231e+1 −	-6.2541e+1 −	-3.3914e+1 −	-4.7807e+1 −	-7.9445e+1
DC-MaTS4	25	25	-2.0450e+1 −	-1.7286e+1 −	-6.0188e+0 −	-1.9663e+1 −	-1.2359e+1 −	-1.7168e+1 −	-2.3322e+1
	50	50	-3.6998e+1 −	-3.1583e+1 −	-1.4475e+0 −	-3.7999e+1 −	-2.2271e+1 −	-3.4333e+1 −	-4.5822e+1
	75	75	-5.2019e+1 −	-4.8166e+1 −	-3.1277e+1 −	-5.4460e+1 −	-3.2265e+1 −	-5.1838e+1 −	-6.7064e+1
	100	100	-6.6035e+1 −	-6.3448e+1 −	-4.1197e+1 −	-6.4686e+1 −	-4.1726e+1 −	-6.7528e+1 −	-8.2948e+1
NMLR	25	10	4.6768e-1 −	5.7095e-1 −	5.2788e-1 −	5.1743e-1 −	5.0437e-1 −	4.3181e-1 −	8.2622e-2
	50	10	1.8806e+0 −	2.2028e+0 −	2.5395e+0 −	1.3811e+0 −	1.7005e+0 −	1.4314e+0 −	4.0024e-1
	75	10	3.2172e+0 −	2.0220e+0 −	3.5594e+0 −	2.3097e+0 −	2.8114e+0 −	2.8416e+0 −	7.1137e-1
	100	10	5.7084e+0 −	3.9656e+0 −	7.2406e+0 −	3.9970e+0 −	5.2747e+0 −	6.0221e+0 −	1.4172e+0
+/−/≈			0/20/0	0/20/0	0/20/0	0/20/0	0/20/0	0/20/0	



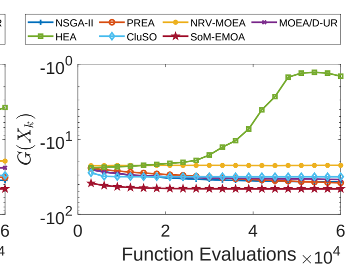
(a) DC-MaTS1



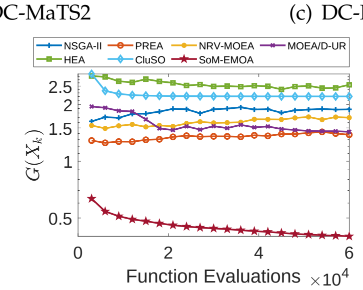
(b) DC-MaTS2



(c) DC-MaTS3



(d) DC-MaTS4

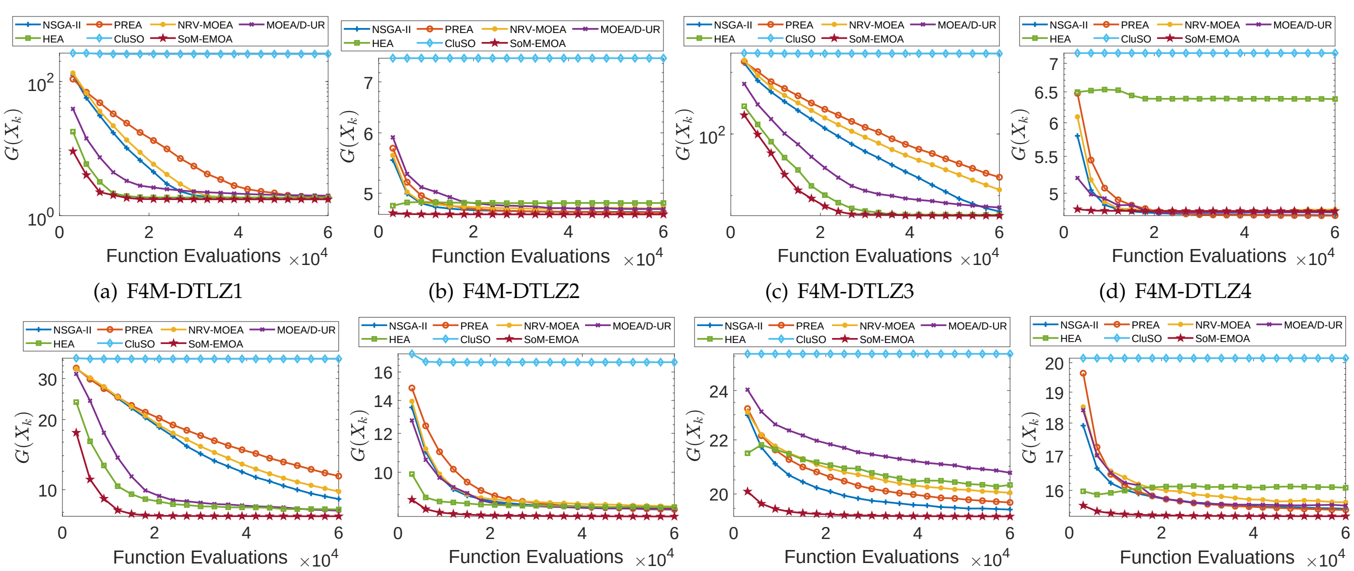


(e) NMLR

Fig. S-1. The convergence curve of different algorithms on 5 existing F4M optimization benchmark problems. The number of objectives $m = 50$ and the size of the solution set $k = 10$.

TABLE S-II
THE MEAN VALUES OF $G_{\mathrm{WS}}(X_k)$ OBTAINED BY SOM-EMOA AND ALL BASELINES ON THE PROPOSED BENCHMARK TEST PROBLEMS. THE SIZE OF THE FINAL SOLUTION SET k IS 10. THE '+', '−' AND '≈' INDICATE THAT THE COMPARED ALGORITHM IS 'SIGNIFICANTLY BETTER THAN', 'SIGNIFICANTLY WORSE THAN' AND 'STATISTICALLY SIMILAR TO' SOM-EMOA, RESPECTIVELY.

Problem	m	d	NSGA-II	PREA	NRV-MOEA	MOEA/D-UR	HEA	CluSO	SoM-EMOA
F4M-DTLZ1	25	7	8.2931e-1 −	1.0842e+2 −	8.5356e-1 −	8.5393e-1 −	8.3737e-1 −	8.4408e-1 −	7.9119e-1
	50	7	1.8374e+0 −	2.5636e+2 −	1.8601e+0 −	1.8688e+0 −	1.8425e+0 −	1.9886e+0 −	1.7520e+0
	75	7	2.8132e+0 −	3.5145e+2 −	2.8425e+0 −	2.8405e+0 −	2.8163e+0 −	3.1695e+0 −	2.7244e+0
	100	7	3.7875e+0 −	4.6900e+2 −	3.8287e+0 −	3.8183e+0 −	3.7985e+0 −	4.3576e+0 −	3.6819e+0
F4M-DTLZ2	25	12	2.2054e+0 −	3.5437e+0 −	2.2643e+0 −	2.2019e+0 −	2.2366e+0 −	2.2349e+0 −	2.1791e+0
	50	12	4.7227e+0 −	7.5198e+0 −	4.8530e+0 −	4.7171e+0 −	4.7828e+0 −	4.7651e+0 −	4.6892e+0
	75	12	7.2174e+0 −	1.1299e+1 −	7.4054e+0 −	7.2045e+0 −	7.2756e+0 −	7.3863e+0 −	7.1707e+0
	100	12	9.7520e+0 −	1.5659e+1 −	9.9627e+0 −	9.7517e+0 −	9.8429e+0 −	1.0002e+1 −	9.6826e+0
F4M-DTLZ3	25	12	2.9316e+0 −	9.5623e+2 −	2.3036e+0 −	9.7777e+0 −	4.9379e+0 −	2.3439e+0 −	2.1929e+0
	50	12	5.5111e+0 −	2.0168e+3 −	4.8720e+0 −	1.9824e+1 −	1.2426e+1 −	6.4009e+0 −	4.7151e+0
	75	12	8.2502e+0 −	3.2007e+3 −	7.4338e+0 −	2.9948e+1 −	1.4539e+1 −	1.2318e+1 −	7.2048e+0
	100	12	1.1595e+1 −	4.2114e+3 −	9.9897e+0 −	4.1432e+1 −	2.0185e+1 −	1.7760e+1 −	9.7253e+0
F4M-DTLZ4	25	12	2.2059e+0 +	3.5730e+0 −	2.3089e+0 −	2.2019e+0 +	2.2389e+0 −	2.2227e+0 −	2.2182e+0
	50	12	4.7221e+0 +	7.1872e+0 −	6.3818e+0 −	4.7171e+0 +	4.7941e+0 −	4.7574e+0 +	4.7742e+0
	75	12	7.2145e+0 −	1.0770e+1 −	8.7160e+0 −	7.2053e+0 −	7.2825e+0 −	7.8706e+0 −	7.1700e+0
	100	12	9.7534e+0 −	1.4704e+1 −	1.1408e+1 −	9.7510e+0 −	9.8346e+0 −	1.0437e+1 −	9.6835e+0
F4M-WFG1	25	12	4.4740e+0 −	1.8141e+1 −	3.9012e+0 −	5.5467e+0 −	4.9297e+0 −	3.8980e+0 −	3.5080e+0
	50	12	9.1162e+0 −	3.6448e+1 −	8.2233e+0 −	1.1436e+1 −	9.8143e+0 −	8.1241e+0 −	7.6892e+0
	75	12	1.3897e+1 −	5.3911e+1 −	1.2480e+1 −	1.7346e+1 −	1.4680e+1 −	1.2908e+1 −	1.1718e+1
	100	12	1.8708e+1 −	7.2710e+1 −	1.6922e+1 −	2.2689e+1 −	2.0147e+1 −	1.7465e+1 −	1.5824e+1
F4M-WFG2	25	12	3.8929e+0 −	7.9554e+0 −	3.9269e+0 −	3.8799e+0 −	3.9581e+0 −	3.8919e+0 −	3.7455e+0
	50	12	8.3892e+0 −	1.6752e+1 −	8.4895e+0 −	8.3951e+0 −	8.5151e+0 −	8.4229e+0 −	8.1254e+0
	75	12	1.2812e+1 −	2.5251e+1 −	1.2997e+1 −	1.2833e+1 −	1.3024e+1 −	1.3070e+1 −	1.2387e+1
	100	12	1.7169e+1 −	3.4546e+1 −	1.7478e+1 −	1.7234e+1 −	1.7485e+1 −	1.7611e+1 −	1.6723e+1
F4M-WFG3	25	12	9.1562e+0 −	1.1933e+1 −	9.4284e+0 −	9.2865e+0 −	9.4911e+0 −	9.4503e+0 −	8.9982e+0
	50	12	1.9457e+1 −	2.5601e+1 −	2.0320e+1 −	1.9702e+1 −	2.0043e+1 −	2.0761e+1 −	1.9222e+1
	75	12	2.9738e+1 −	3.8368e+1 −	3.0877e+1 −	2.9995e+1 −	3.0523e+1 −	3.2067e+1 −	2.9386e+1
	100	12	3.9970e+1 −	5.1698e+1 −	4.1395e+1 −	4.0342e+1 −	4.1038e+1 −	4.3674e+1 −	3.9639e+1
F4M-WFG4	25	12	7.3034e+0 −	9.5019e+0 −	7.4554e+0 −	7.2932e+0 −	7.4153e+0 −	7.3712e+0 −	7.1506e+0
	50	12	1.5497e+1 −	2.0161e+1 −	1.6077e+1 −	1.5465e+1 −	1.5672e+1 −	1.5582e+1 −	1.5294e+1
	75	12	2.3892e+1 −	3.1005e+1 −	2.4410e+1 −	2.3828e+1 −	2.4082e+1 −	2.4354e+1 −	2.3522e+1
	100	12	3.2160e+1 −	4.1648e+1 −	3.2785e+1 −	3.2096e+1 −	3.2447e+1 −	3.2879e+1 −	3.1790e+1
+/−/≈			2/30/0	0/32/0	0/32/0	2/30/0	0/32/0	1/31/0	



(a) F4M-DTLZ1 (b) F4M-DTLZ2 (c) F4M-DTLZ3 (d) F4M-DTLZ4
(e) F4M-WFG1 (f) F4M-WFG2 (g) F4M-WFG3 (h) F4M-WFG4

Fig. S-2. The convergence curve of different algorithms on the proposed F4M optimization benchmark problems. The number of objectives $m = 50$ and the size of the solution set $k = 10$.

S-III. Comparison of NSGA-II and SoM-EMOA under the same runtime budget

NSGA-II is much (roughly 10 times) faster than SoM-EMOA. The question is: given the same runtime, will NSGA-II outperform SoM-EMOA on some problems?

To answer the question, we perform an experiment by increasing the number of FEs from 60,000 to 600,000 for NSGA-II and keep FEs to 60,000 for SoM-EMOA, since roughly NSGA-II is 10 times faster than SoM-MOEA. Then we compare the performance of NSGA-II and SoM-EMOA under this setting. The experimental results are shown in Table S-III. From the results we can see that SoM-EMOA outperforms NSGA-II on 47 out of 52 test instances. NSGA-II only performs better than SoM-EMOA on 3 instances. This shows that even given the same runtime, SoM-EMOA still outperforms NSGA-II in most cases.

References

[1] Y. Liu, C. Lu, X. Lin, and Q. Zhang, "Many-objective cover problem: Discovering few solutions to cover many objectives," in *International Conference on Parallel Problem Solving from Nature*. Springer, 2024, pp. 68–82.

[2] X. Lin, Y. Liu, X. Zhang, F. Liu, Z. Wang, and Q. Zhang, "Few for many: Tchebycheff set scalarization for many-objective optimization," *arXiv preprint arXiv:2405.19650*, 2024.

TABLE S-III
THE MEAN VALUES OF $G_{\mathrm{WS}}(X_k)$ OBTAINED BY NSGA-II AND SOM-EMOA WHERE 600,000 FES ARE USED IN NSGA-II AND 60,000 FES ARE USED IN SOM-EMOA. THE SIZE OF THE FINAL SOLUTION SET k IS 5. THE '+', '−' AND '≈' INDICATE THAT NSGA-II IS 'SIGNIFICANTLY BETTER THAN', 'SIGNIFICANTLY WORSE THAN' AND 'STATISTICALLY SIMILAR TO' SOM-EMOA, RESPECTIVELY.

Problem	m	d	NSGA-II	SoM-EMOA
NMLR	25	10	1.7126e+0 −	2.0673e-1
	50	10	5.6812e+0 −	7.9513e-1
	75	10	1.3804e+1 −	2.1562e+0
	100	10	1.7767e+1 −	3.7980e+0
DC-MaTS1	25	25	-1.6296e+1 −	-2.0761e+1
	50	50	-3.1408e+1 −	-3.9372e+1
	75	75	-4.5796e+1 −	-5.4906e+1
	100	100	-6.0097e+1 −	-6.9271e+1
DC-MaTS2	25	25	-1.6520e+1 −	-2.0725e+1
	50	50	-3.1124e+1 −	-4.0098e+1
	75	75	-4.5763e+1 −	-5.8179e+1
	100	100	-6.0309e+1 −	-7.4708e+1
DC-MaTS3	25	25	-1.6323e+1 −	-2.0761e+1
	50	50	-3.0949e+1 −	-3.8744e+1
	75	75	-4.4859e+1 −	-5.0979e+1
	100	100	-5.8577e+1 −	-6.1623e+1
DC-MaTS4	25	25	-1.6906e+1 −	-2.0758e+1
	50	50	-3.1991e+1 −	-4.0054e+1
	75	75	-4.5936e+1 −	-5.4265e+1
	100	100	-6.0442e+1 −	-6.6698e+1
F4M-DTLZ1	25	7	1.0009e+0 −	9.4407e-1
	50	7	2.1073e+0 −	2.0601e+0
	75	7	3.2237e+0 −	3.1594e+0
	100	7	4.3375e+0 −	4.2456e+0
F4M-DTLZ2	25	12	2.3828e+0 −	2.3770e+0
	50	12	5.0235e+0 −	5.0055e+0
	75	12	7.6830e+0 −	7.6692e+0
	100	12	1.0350e+1 −	1.0316e+1
F4M-DTLZ3	25	12	2.3820e+0 +	2.3900e+0
	50	12	5.0243e+0 ≈	5.0234e+0
	75	12	7.6835e+0 +	7.6933e+0
	100	12	1.0347e+1 ≈	1.0352e+1
F4M-DTLZ4	25	12	2.3838e+0 −	2.3773e+0
	50	12	5.0240e+0 −	5.0043e+0
	75	12	7.6857e+0 −	7.6689e+0
	100	12	1.0350e+1 +	1.0476e+1
F4M-WFG1	25	12	4.3869e+0 −	4.3270e+0
	50	12	9.2102e+0 −	9.0636e+0
	75	12	1.3972e+1 −	1.3769e+1
	100	12	1.8762e+1 −	1.8475e+1
F4M-WFG2	25	12	4.6329e+0 −	4.5091e+0
	50	12	9.7560e+0 −	9.6015e+0
	75	12	1.4829e+1 −	1.4602e+1
	100	12	1.9873e+1 −	1.9637e+1
F4M-WFG3	25	12	9.9804e+0 −	9.8869e+0
	50	12	2.1342e+1 −	2.1094e+1
	75	12	3.2636e+1 −	3.1778e+1
	100	12	4.3977e+1 −	4.2688e+1
F4M-WFG4	25	12	7.6724e+0 −	7.6640e+0
	50	12	1.6344e+1 −	1.6294e+1
	75	12	2.4933e+1 −	2.4913e+1
	100	12	3.3497e+1 −	3.3445e+1
+/ − / ≈			3/47/2	