Supplementary Material for "Time-Varying Formation for Second-Order Open Multi-Agent Systems With External Disturbances"

Min Gong, Yuanqing Xia and Yuezu Lv

Appendix A
Proof of Lemma 2

Proof. Let $e_{\kappa_i}(t) = [e_{v,\kappa_i}(t), e_{d,\kappa_i}(t)]^T$ be the error of the observer, where $e_{v,\kappa_i}(t) = v_{\kappa_i}(t) - z_{\kappa_i 0}(t)$ and $e_{d,\kappa_i}(t) = d_{\kappa_i}(t) - z_{\kappa_i}(t)$, we have

$$\begin{aligned}\dot{e}_{\kappa_i}(t) &= \begin{bmatrix} -\beta_1 & 1 \\ -\beta_2 & 0 \end{bmatrix} e_{\kappa_i}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \dot{d}_{\kappa_i}(t) \\ &= A_e e_{\kappa_i}(t) + B_2 \dot{d}_{\kappa_i}(t).\end{aligned} \tag{A.1}$$

Since $\beta_1 > 0$ and $\beta_2 > 0$, A_e is Hurwitz. For any given symmetric positive definite matrix Q, there exists a symmetric positive definite matrix P_1 that satisfies

$$A_e^T P_1 + P_1 A_e + Q = 0. \tag{A.2}$$

Choose the following Lyapunov function

$$V_{1\kappa_i}(t) = e_{\kappa_i}^T(t) P_1 e_{\kappa_i}(t). \tag{A.3}$$

The derivative of Lyapunov function is

$$\begin{aligned}\dot{V}_{1\kappa_i}(t) &= e_{\kappa_i}^T(t)(P_1 A_e + A_e^T P_1) e_{\kappa_i}(t) + 2 e_{\kappa_i}^T(t) P_1 B_2 \dot{d}_{\kappa_i}(t) \\ &\leq -e_{\kappa_i}^T(t) Q e_{\kappa_i}(t) + 2\|P_1 B_2\| \, \|e_{\kappa_i}(t)\| \, \|\dot{d}_{\kappa_i}(t)\| \\ &\leq -\lambda_{\min}(Q) \|e_{\kappa_i}(t)\|^2 + 2J\|P_1 B_2\| \, \|e_{\kappa_i}(t)\|.\end{aligned} \tag{A.4}$$

where $\lambda_{\min}(Q)$ is the minimum eigenvalue of Q. It can be concluded that the observation error has an upper bound $(2J\|P_1B_2\|)/\lambda_{\min}(Q)$. This completes the proof.

Appendix B
Proof of Theorem 1

Proof. Consider the following Lyapunov function

$$V_2(t) = \tilde{\xi}_{F_\kappa}^T(t) \left(\Gamma_\kappa \otimes P_2\right) \tilde{\xi}_{F_\kappa}(t), \tag{B.1}$$

where Γ_κ is defined in inequality (2). Since the topology of the system (3) is invariant at time intervals $[t_\kappa, t_{\kappa+1})$ $(\kappa = 1, 2, \cdots)$, the function $V_2(t)$ is piecewise continuous and can be differentiated over time interval $[t_\kappa, t_{\kappa+1})$, i.e.,

$$\begin{aligned}\dot{V}_2(t) =& \tilde{\xi}_{F_\kappa}^T(t) \left[\Gamma_\kappa \otimes \Omega_B + \bar{\Omega}_L\right] \tilde{\xi}_{F_\kappa}(t) \\ &+ 2\tilde{\xi}_{F_\kappa}^T(t) \left(\Gamma_\kappa \otimes P_2^{-1} B_2\right) e_{d,F_\kappa}(t),\end{aligned} \tag{B.2}$$

where $\Omega_B = P_2\left(B_1 B_2^T + B_2\alpha\right) + \left(B_1 B_2^T + B_2 \alpha\right)^T P_2$, $\bar{\Omega}_L = -\delta[(L_\kappa^{\mathrm{ff}})^T \Gamma_\kappa + \Gamma_\kappa L_\kappa^{\mathrm{ff}}] \otimes P_2 B_2 B_2^T P_2$. According to (2), (10) and Young's Inequality, we have

$$\begin{aligned}\dot{V}_2(t) \leq& \tilde{\xi}_{F_\kappa}^T(t) \left[(2-2\delta\gamma_\kappa)\,\Gamma_\kappa \otimes P_2 B_2 B_2^T P_2 - \Gamma_\kappa \otimes \varepsilon P_2\right] \tilde{\xi}_{F_\kappa}(t) \\ &+ e_{d,F_\kappa}^T(t) \left(\Gamma_\kappa \otimes I_2\right) e_{d,F_\kappa}(t).\end{aligned} \tag{B.3}$$

Since $\delta > 1/\bar{\gamma}$, based on Lemma 3, one has

$$\begin{aligned}\dot{V}_2(t) \leq& -\varepsilon \tilde{\xi}_{F_\kappa}^T(t) \left[\Gamma_\kappa \otimes P_2\right] \tilde{\xi}_{F_\kappa}(t) + \left(\Gamma_\kappa \otimes I_2\right) \|e_{d,F_\kappa}(t)\|^2 \\ \leq& -\varepsilon V_2(t) + \Pi,\end{aligned} \tag{B.4}$$

where $\Pi = \max_\kappa\{\Pi_\kappa\}$. Integrate (18) during $[t_\kappa^+, t)$ and $[t_{\kappa-1}^+, t_\kappa^-)$ respectively, we can obtain

$$V_2(t) \leq \left(V_2\left(t_\kappa^+\right) - \frac{\Pi}{\varepsilon}\right) e^{-\varepsilon\left(t - t_\kappa^+\right)} + \frac{\Pi}{\varepsilon}, \tag{B.5}$$

$$V_2(t_\kappa^-) \leq \left(V_2\left(t_{\kappa-1}^+\right) - \frac{\Pi}{\varepsilon}\right) e^{-\varepsilon\left(t_\kappa^- - t_{\kappa-1}^+\right)} + \frac{\Pi}{\varepsilon}. \tag{B.6}$$

According to (9), we can get that

$$V_2(t_\kappa^+) \leq \mu V_2(t_\kappa^-) + \Delta \tag{B.7}$$

hold at switching time t_κ. Thus, the following inequality holds

$$\begin{aligned}V_2(t) \leq& \left(V_2\left(t_\kappa^+\right) - \frac{\Pi}{\varepsilon}\right) e^{-\varepsilon\left(t - t_\kappa^+\right)} + \frac{\Pi}{\varepsilon} \\ \leq& \cdots \\ \leq& \left(V_2(0) - \frac{\Pi}{\varepsilon}\right) \mu^{\kappa-1} e^{-\varepsilon t} + \frac{\Pi}{\varepsilon} \\ &+ \left[(\mu - 1)\frac{\Pi}{\varepsilon} + \Delta\right] \sum_{\sigma=2}^{\kappa} \mu^{\kappa-\sigma} e^{\varepsilon\left(t_\sigma^+ - t\right)} \\ \leq& \left(V_2(0) - \frac{\Pi}{\varepsilon}\right) \mu^{N_0} e^{\left(\frac{1}{\varsigma}\ln\mu - \varepsilon\right)t} + \frac{\Pi}{\varepsilon} \\ &+ \left[(\mu - 1)\frac{\Pi}{\varepsilon} + \Delta\right] \frac{e^{\varepsilon N_0 \varsigma}\left[1 - e^{(\ln\mu - \varepsilon\varsigma)(\kappa-1)}\right]}{1 - e^{\ln\mu - \varepsilon\varsigma}},\end{aligned} \tag{B.8}$$

which implies that $\tilde{\xi}_{F_\kappa}(t)$ converges to $\mathfrak{U}$ as $t \to \infty$. This completes the proof.